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#### Abstract

We investigate the well-known heuristic due to Warnsdorff for finding knight's tours on chessboards. This heuristic is incomplete because it does not provide a tiebreaking rule. Our computer experiments confirm the conventional wisdom that arbitrary tiebreaking is likely to produce a tour on a square board of dimension smaller than 50. But our experiments strongly suggest that for larger boards, the probability that random tiebreaking will produce a tour rapidly goes to 0.

We investigated a tiebreak method involving move-orderings and found that for each of the  $8!$  such move-orderings, there was a board on which this move-ordering failed to give a tour. On the other hand, on each small to medium size board some move-ordering could be used to find a tour; for example, nineteen different move-orderings sufficed to find tours on rectangular boards of dimension between 50 and 300.

Finally, we discovered and we partially prove that by combining different move-orderings in different regions of the board one can achieve a tiebreaking rule which produces tours on all square boards of dimension larger than 112. Thus we convert Warnsdorff's heuristic into an algorithm.

# A Warnsdorff-Rule Algorithm for Knight's Tours on Square Chessboards

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## 1 Introduction

A knight on a chessboard can move from a given square to any of eight squares; in Figure 1, the knight is in the square marked with an x and the eight possible moves are numbered. We will call each of these eight squares *adjacent* to the original square. A *knight's path* is a sequence of squares, each adjacent to the next, such that no square appears in the sequence more than once. A *knight's tour* is a knight's path which includes every square of the chessboard. We will depict knight's paths and tours on an  $m$  by  $n$  chessboard using a matrix like this (here  $m$  and  $n$  are both 8):

1	16	27	22	3	18	47	56
26	23	2	17	46	57	4	19
15	28	25	62	21	48	55	58
24	35	30	45	60	63	20	5
29	14	61	34	49	44	59	54
36	31	38	41	64	53	6	9
13	40	33	50	11	8	43	52
32	37	12	39	42	51	10	7

Here the squares are labelled in the order they appear in the sequence. If this were not a tour, then squares not included in the sequence would be labelled with a dash ("-"). A square of the matrix will be denoted by an ordered pair giving its row number (from 1 to  $m$ ), followed by its column number (from 1 to  $n$ ).

Finding a knight's tour is a special case of the NP-hard problem of finding a Hamiltonian path on a generic graph [4] and because it can be easily solved, its study provides a method of understanding the more difficult general problem.

Knight's tours are known to exist on any  $m$  by  $n$  chessboard with both  $m$  and  $n$  larger than 4, and algorithms to produce them are also known [2, 3, 8]. Each such solution uses a divide-and-conquer method, in which tours on certain small

	8		1	
7				2
		X		
6				3
	5		4	

Figure 1: Legal knight's moves.

boards are patched together to produce a large tour. Each of the algorithms given runs in time  $O(mn)$ .

## 2 Warnsdorff's Rule

In 1823 Warnsdorff [11] proposed a simple rule which he suggested gave a tour on the 8 by 8 chessboard.

**Warnsdorff's Rule.** Construct a knight's path on an  $m$  by  $m$  board as follows: Let the  $(n + 1)$ th square of the path be the square which

1. is adjacent to the  $n$ th square,
2. is unvisited (that is, it does not appear earlier in the path), and
3. has the minimal number of adjacent, unvisited squares.

Should more than one square satisfy these three conditions, the tie may be broken at random; should no square satisfy the conditions, the path terminates.

Clearly this rule is not deterministic; Warnsdorff claimed that no matter which random choices are made to break ties, the path produced is always a tour.

In fact, on the 8 by 8 board or other small boards, if several incorrect choices are made a path can be produced which is not a tour. For example, with the appropriate choices the following path can be produced on the 5 by 5 board:

11	6	1	-	13
-	-	12	7	2
5	10	-	14	-
-	-	16	3	8
17	4	9	-	15

However, cases like this one are rare on small boards, and when  $m$  is less than about 50 it is easy to find a tour by repeating the application of the rule until a tour is found.

Unfortunately, on larger boards the rule becomes rapidly less useful. For example, Parberry [8] used a variant of Warnsdorff's rule with random tiebreaks: after producing a path like the one above which missed some squares, he applied Warnsdorff's rule (again with random tiebreaks) to the remaining squares. He then repeated this process to produce a number of paths on subsets of the board; his algorithm then attempted to piece these smaller paths together to produce a tour of the full board. The whole process was tried repeatedly until a tour was found. However, Parberry reports that the largest board for which this method succeeded was the 78 by 78 board, and his experiments show an exponential decrease in success rate as the size of the board increases.

Our own experiments also confirm this observation. For each  $m$  with  $5 \leq m \leq 400$  we made 100 trials of Warnsdorff's rule with random tiebreaks on an  $m$  by  $m$  board. The results appear in Figure 2; the vertical axis gives the number of tours obtained from 100 trials as a function of  $m$ . As expected, the rule is very successful for small boards, succeeding more than 50% of the time for almost all  $m < 100$ . However, the success rate drops sharply as  $m$  increases; when  $m > 200$  we find that fewer than 5% of the attempts produce tours, and when  $m > 325$  we find no successes at all. These observations provide strong evidence that the success rate of a random tiebreak method goes rapidly to 0 as  $m$  increases.

However, it appears that the rule can be improved by replacing the random selection used with a better tiebreaking rule. For example, Pohl [9] used the following tiebreak rule: if  $S_1, S_2, \dots, S_k$  are the squares which satisfy the three conditions of Warnsdorff's rule, assign to each  $S_i$  the number of unvisited squares adjacent to  $S_i$  and select a  $S_i$  whose score is the lowest. Any remaining ties are broken using a fixed ordering of moves (see below). Pohl reports some success with this rule, finding tours on many starting squares with the 8 by 8 chessboard and finding a 20 by 20 and 40 by 40 tour. However, we tested the rule and found that it was possible for it to fail, although it was more successful than the ordinary Warnsdorff's rule.

To get another tiebreak rule, we define a *move-ordering* to be an ordering of the set  $\{1, 2, 3, 4, 5, 6, 7, 8\}$ . We will depict such orderings as lists of these numbers, for example 54271683, where the leftmost number is the first in the ordering and the rightmost is the last. These move-orderings correspond to simple tiebreaking rules as follows: number the possible moves of a knight as indicated in Figure 1, and if a choice is presented among several possible moves, select the one which appears first in the move-ordering. For example, the 8 by 8 tour presented in the introduction was produced by the tiebreaking rule 12345678, while the unsuccessful path on the 5 by 5 board presented above is produced by the tiebreaking rule 36875124.

One might hope that some move-ordering might succeed in producing tours

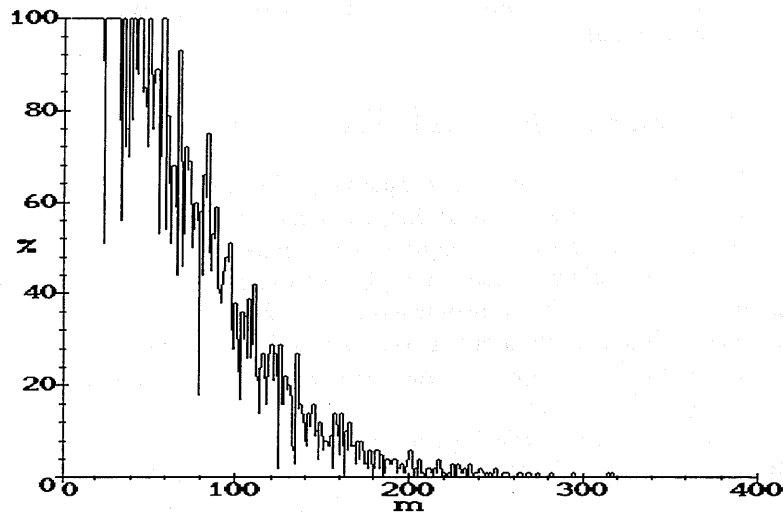


Figure 2: Success rate of random tiebreak rule.

on all boards. However, our computer experiments dash this hope. We tried each of the  $8! = 40320$  possible move-orderings on each of the  $m$  by  $n$  boards with  $10 \leq m, n \leq 50$  and found that each move-ordering failed on at least one such board. Thus the move-ordering method is unsuccessful even for small boards.

However, we obtained a surprising result when we tried a number of move-orderings in succession. On each of the  $m$  by  $n$  boards with  $10 \leq m, n < 300$ , we tried a sequence of move-orderings produced by an algorithm in Nijenhuis and Wilf [7] which produces all permutations of  $k$  objects in a fixed order without repeating any. We expected that for many boards, a large number of trials would be needed before a tour was found, and hence that the program would be unworkably slow. On the contrary, on every such board one of the first nineteen of the move-orderings produced by the algorithm gave a tour. Hence this list of nineteen move-orderings suffices to give tours on every such board; one simply tries each of them in succession. We present these nineteen move-orderings below.

12345678, 21345678, 31245678, 13245678, 23145678, 32145678, 42135678,  
 24135678, 14235678, 41235678, 21435678, 12435678, 13425678, 31425678,  
 41325678, 14325678, 34125678, 43125678, 43215678

It is not at all clear however that this list of move-orderings, or another short list, would suffice for all boards. Therefore in the next section we turn to another method of tiebreaking.

### 3 A Successful Tiebreak Rule

We now combine move-orderings in a simple way. We apply a certain move-ordering to all ties that appear in our path, until a certain predetermined square is reached. Any tie appearing at this square, and those following, is broken by using a second move-ordering, which we apply until another square is reached. Here we again switch to another move-ordering, and so on.

We were able to find a method of this type which is successful for producing tours on all square boards; we present the method in this section and prove an example case in the next section.

Our combination of move-orderings is presented in Figure 3. Each of our tours begins in the upper left-hand corner. There are eight different combinations to be used, according to the value of  $m \bmod 8$  (in one case, a switching square must be adjusted according to the value of  $m \bmod 16$ , so there are really nine cases). The first column of the table gives the move-orderings to be used, in order; the second column gives the squares at which one should switch to the next move-ordering. So, for example, if  $m \equiv 1 \pmod 8$ , we begin in the upper left-hand corner  $(1,1)$  and apply the move-ordering 34261578 until the square  $(m-1, m-2)$  is reached. Ties appearing at that square and those following should be broken using the move-ordering 87642135, until the square  $(2,2)$  is reached; then the tiebreak rule 51324678 is used until the square  $(m-6, (m-1)/2+5)$  is reached, and from this point forward, the move-ordering 32481765 is used, until eventually a dead-end is reached.

The compact description provided by Warnsdorff's Rule permits us to write down an explicit algorithm for producing these tours.

**Algorithm.** Given an integer  $m \geq 112$ , this algorithm uses Warnsdorff's rule together with a certain tiebreak rule to produce a knight's tour on the  $m$  by  $m$  chessboard. We use  $m$  by  $m$  matrices  $A$  and  $B$ , an vector  $K$  of 8 ordered pairs, a vector  $T$  of 8 integers (used to hold move-orderings) and integers  $c, i, j, k, t, u, v, x, x', x'', y, y', y''$ , and  $z$ .

1. [Initialize] For  $i = 1, \dots, m$  and  $j = 1, \dots, n$ , set  $A[i, j] := -1$  and  $B[i, j] := 0$ . Set  $K := [(-2, 1), (-1, 2), (1, 2), (2, 1), (2, -1), (1, -2), (-1, -2), (-2, -1)]$ . Set  $c := 1, t := 0, (u, v) := (1, 1), (x, y) := (1, 1)$ .
2. [Initialize matrices] For  $i = 1, \dots, m$  and for  $j = 1, \dots, n$ , do the following: for each  $k = 1, \dots, 8$ , set  $(x', y') := (i, j) + K[k]$ , and if  $1 \leq x' \leq m$  and  $1 \leq y' \leq n$ , set  $B[i, j] := B[i, j] + 1$ .
3. [Do tour] Set  $A[x, y] := c, c := c + 1, z := 9$ , and  $k := 1$ .

$m \equiv 0 \pmod{8}$	
34261578	$(m - 1, m - 2)$
87642135	$(2, 2)$
51867342	$(m - 8, 1)$
51342678	$(7, m - 3)$
21435678	end
$m \equiv 1 \pmod{8}$	
34261578	$(m - 1, m - 2)$
87642135	$(2, 2)$
51324678	$(m - 6, (m - 1)/2 + 5)$
32481765	end
$m \equiv 2 \pmod{8}$	
34261578	$(6, 1)$
87642135	$(3, 1)$
54132678	$(m - 15, 4)$
52431678	$(10, m - 2)$
85647123	$(5, m/2 - 3)$
15746823	end
$m \equiv 3 \pmod{8}$	
34625718	$(m - 1, m - 2)$
42681357	$(m - 6, m)$
86512347	$(2, 5)$
51867342	$(m - 10, 3)$
61825437	$((m - 1)/2 + 1, m - 2)$
71642538	end
$m \equiv 4 \pmod{8}$	
34261578	$(m - 1, m - 2)$
87642135	$(2, 2)$
51867342	$(m - 8, 1)$
51342678	$(10, m - 5)$
86753421	$(13, m/2 + 1)$
78563421	end
$m \equiv 5 \pmod{8}$	
34261578	$(m - 1, m - 2)$
87642135	$(2, 2)$
51324678	*
15234678	end
$m \equiv 6 \pmod{8}$	
34261578	$(6, 1)$
87642135	$(3, 1)$
54132678	$(m - 10, 1)$
52431678	$(10, m - 2)$
85647123	$(3, m/2 + 4)$
12453678	end
$m \equiv 7 \pmod{8}$	
34625718	$(m - 1, m - 2)$
42681357	$(m - 6, m)$
86512347	$(2, 5)$
51867342	$(m - 6, 3)$
61825437	$((m - 1)/2 + 1, m - 2)$
61357284	end

\* If  $m \equiv 5 \pmod{16}$ , use  $(m - 2, (m - 1)/2 - 2)$ , else use  $(m - 2, (m - 1)/2 - 6)$ .

Figure 3: Summary of tiebreaking rules.

4. [Change T if necessary] If  $(x, y) = (u, v)$ , set  $t := t + 1$ . Set  $T$  to the  $t$ th of the move-orderings appearing in Figure 3 in the section corresponding to the value of  $m \bmod 8$ . Set  $(u, v)$  to the  $t$ th of the switching squares appearing in the same section (if  $m \equiv 5 \pmod{8}$  and  $t = 3$ , set  $(u, v)$  to the appropriate square according to the value of  $m \bmod 16$ , while if this is the last move-ordering, set  $(u, v) := (-1, -1)$ ).
5. [Find minimal degree move] Set  $(x', y') = (x, y) + K[T[k]]$ . If  $1 \leq x \leq m$  and  $1 \leq y \leq n$  and  $A[x', y'] = -1$ , set  $B[x', y'] := B[x', y'] - 1$  and, if  $B[x', y'] < z$ , set  $(x'', y'') := (x', y')$  and  $z := B[x', y']$ . Set  $k := k + 1$  and if  $k \leq 8$  go to step 5.
6. [Proceed to next square] If  $z = 9$  then go to step 7, else set  $(x, y) = (x'', y'')$  and go to step 3.
7. Output  $A$  as the knight's tour matrix and terminate the algorithm.

## 4 Form Matrix Notation

Our proof of the algorithm's correctness is constructive, and hence requires that we give instructions for constructing knight's paths on chessboards of unbounded size. We present a matrix notation, which we call the *form matrix*, which allows us to do this without introducing very large matrices.

A form matrix contains letters, numbers, dashes, and lines, and provides a recipe for producing a knight's path on any  $m$  by  $m$  board where  $m$  is larger than some bound and satisfies a given congruence condition (usually  $\bmod 4$  or  $\bmod 8$ ). Several form matrices are used in the next section to give a tour in successive stages; the method of connecting the paths given by each form matrix will be described in each case.

Dashes, as above, indicate that a square is unvisited at this stage of the path and will also not be visited during this stage. On the other hand, the special letter "x" indicates that a square has been visited at an earlier stage (the word "letter" from here on will exclude "x".)

Letters and numbers are used to produce the given path. We give letters the alphabetic ordering and therefore speak of "a" as "smaller" than "b", etc.

We are always given a starting square, which will contain an "a" or a "1". When the current square contains a number, we progress always to an unvisited square containing the next higher number, and when no adjacent unvisited square contains the next higher number we move to the smallest letter available. When the current square contains a letter, we move if possible to an unvisited square containing the same letter; failing that we move to an unvisited square containing the next largest letter, and if this too is impossible we move to the unvisited square containing the lowest number. If at any point the current



square is not adjacent to any unvisited square containing a letter or number, the path terminates.

It may seem that these rules could fail to determine the path, for instance if two a's are both adjacent to a square also containing an "a"; however we will arrange our form matrices so that this cannot happen.

Let us examine an example to illustrate the process above.

a	a	c	7	12	a	c	45	78	a	c	-	-	a	e	-	-	a	c																			
c	8	a	a	1	46	39	a	e	102	-	-	a	e	-	-	a	e	-																			
a	c	6	11	38	13	44	77	66	79	-	-	-	-	-	-	-	-	e	a																		
9	c	15	2	47	40	65	42	101	-	-	-	-	-	-	-	-	-	-	-																		
b	5	10	37	14	43	76	67	80	-	-	-	-	-	-	-	-	-	-	a	e																	
c	16	3	48	21	64	41	100	-	-	-	-	-	-	-	-	-	-	-	-	-																	
4	b	22	19	36	75	68	81	-	-	-	-	-	-	-	-	-	-	-	e	a																	
17	c	49	26	63	20	99	74	-	-	-	-	-	-	-	-	-	-	-	-	-																	
b	23	18	35	50	69	82	-	-	-	-	-	-	-	-	-	-	-	-	-	a	e																
c	34	25	62	27	98	73	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-																
24	b	32	51	70	83	-	-	-	-	-	-	-	-	-	-	-	-	-	-	e	a																
33	c	61	28	97	72	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-																
b	31	52	71	84	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	a	e															
c	60	29	96	53	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-															
30	b	58	85	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	e	a															
59	c	95	54	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-															
b	57	86	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	a	e														
c	94	55	-	87	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-														
56	b	92	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	e	a													
93	c	-	88	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-													
b	91	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	a	e												
c	-	89	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-												
90	b	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	e	a											
-	c	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-											
b	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	a	e										
c	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-										
-	b	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	e	a									
-	c	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-									
b	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	a	e								
c	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-								
-	b	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	d	e	a						
-	c	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	d	a					
b	c	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	e	a	d	a			
c	-	d	b	-	-	d	b	-	-	d	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	d	a		
-	b	c	-	d	b	-	-	d	b	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	a	a

This matrix specifies a knight's path on any  $m$  by  $m$  board with  $m \equiv 3 \pmod{4}$  and  $m \geq 35$ ; the path begins on the upper left-hand corner of the board, (1, 1).

The square (1, 1) contains an "a", and there is a unique square adjacent to this one with an "a" in it, namely (2, 3). Thus the first square of the sequence is (1, 1) and the second is (2, 3). Each successive square with an "a" is adjacent to a unique unvisited square with another 1, and so we obtain the sequence

$$(1, 1), (2, 3), (3, 1), (1, 2), (2, 4), (1, 6), (2, 8), (1, 10).$$

At this point we encounter the vertical line between the 11th and 12th columns. This line plays the role of points of ellipsis, in the sense that it instructs us to continue the pattern we have established in the last few moves. If we use the notation indicated in Figure 1, we see that we have been alternating directions: we first move in the 2 direction and then in the 3 direction. The line we have encountered tells us to continue this pattern as long as necessary to reach a column which is as far from the rightmost column of our board as the 12th column is from the rightmost column of the form matrix; namely, to the  $(m - 7)$ th column. It is easy to see that in fact we will land on the square  $(2, m - 7)$  just as indicated (but only if  $m \equiv 3 \pmod{4}$  as stated), which allows us to continue along the sequence of a's.

Notice that vertical and horizontal lines apply only when we are following a sequence of letters; numbers are not affected by encountering lines. Patterns will always be as simple or simpler than the one presented here; usually they will consist merely of motion in a single direction.

Further, observe that in this example the patterns on opposite sides of the line match, so that if no line were present we would could pass from one side to the other while remaining on the a's. In general we will not require this, since it will be obvious where the pattern is picked up again; therefore the reader should consider the parts of the matrix on either side of any line to be independent. See the first form matrix of the next section for an example of this.

Now we can continue along the a's until we reach the upper right-hand corner and begin moving downward to encounter the horizontal line. The sequence up to this point is as follows:

$$\begin{aligned} &(1, 1), (2, 3), (3, 1), (1, 2), (2, 4), (1, 6), (2, 8), (1, 10), \dots, \\ &(1, m - 9), (2, m - 7), (1, m - 5), (2, m - 3), (1, m - 1), \\ &(3, m), (5, m - 1), (7, m), (9, m - 1), \dots \end{aligned}$$

The horizontal line again tells us to continue our pattern, which here alternates between directions 4 and 5.

We can now continue in this fashion until we reach the last "a", which appears in square  $(m, m - 1)$ . Here we have a unique adjacent square containing a "b", which we make the next square of the path; we may then continue along the squares containing b's just as we did with the a's. (If we look back at square  $(m - 2, m - 1)$ , we can see why this change from "a" to "b" was necessary; otherwise at that point we would have had two available a's from which to choose.) Eventually we reach the final "b" in square  $(5, 1)$ , and proceed to the unique adjacent "c". Continuing in this way, we ultimately arrive at the final "e", on square  $(2, 5)$ , which is not adjacent to any letter.

From this square we progress to the 1 found in square  $(2, 5)$ , then to the 2 found in square  $(4, 4)$ , and so on until we reach the 102 in square  $(2, 10)$ . Here we terminate the path.

To assist the reader, we provide in Figure 5 the result of applying this process to the 39 by 39 board.

## 5 Proof of Algorithm Correctness

In this section we present a partial proof of the algorithm's correctness for a single case, namely  $m \equiv 7 \pmod{8}$ . The proof will proceed by constructing the knight's tour, making use of the form matrix notation developed in the previous section. It will be left to the reader to verify that the sequence produced obeys Warnsdorff's rule with the appropriate tiebreak rules as indicated in Figure 3.

## 5.1 Step 1

In fact we have already presented the first step of the proof, since the form matrix used in the previous section to illustrate our notation also gives the first segment of the path. Observe that the path thus constructed first passes around the edge of the board, then begins to fill the upper left-hand corner.

## 5.2 Step 2

We now present an inductive argument which will allow us to fill a triangle-shaped region on the left side of the board. Suppose that for some  $n$ , there is a  $k \equiv 5 \pmod{8}$  such that if  $(i, j)$  is a square in the first  $n$  terms of the sequence with  $i, j < m - 6$ , then  $i$  and  $j$  satisfy one or more of the following conditions, and suppose further that all such  $(i, j)$  which satisfy at least one of the conditions have been visited.

1.  $i = 1, j \equiv 1$  or  $j \equiv 2 \pmod{4}$
2.  $i = 2, j \equiv 0$  or  $j \equiv 3 \pmod{4}$
3.  $j = 1, i \equiv 1$  or  $i \equiv 2 \pmod{4}$
4.  $j = 2, i \equiv 0$  or  $i \equiv 3 \pmod{4}$
5.  $i \leq k - 2(j - 1)$
6.  $i = k - 2(j - 1) + 1$  and  $j < (k - 5)/2$
7.  $(i, j) \in \{(k, 2), (k - 2, 3), (k + 1, 3), (k - 1, 4), (k - 3, 5)\}$

Suppose further that the  $n$ th square in the sequence is  $(2, (k - 1)/2)$ .

The first four conditions ensure that the border away from the corner looks like the border of the first of our example matrices in the previous section.

Condition 5 is satisfied if and only if the following test succeeds. Place a knight on the square  $(k, 1)$  and repeat the move numbered 1 until the edge is reached; call the set of squares landed on  $D$ . The test is passed for a square  $(i, j)$  if and only if for some  $(a, b)$  in  $D$ , we have  $i \leq a$  and  $j = b$ .

Conditions 6 and 7 ensure that certain squares outside the "wavefront"  $D$  have been visited; the condition on the  $n$ th square ensures that the addition we are about to make to the sequence can begin on the proper square.

It is easy to check that after step 1 above, our path satisfies the conditions given with  $k = 21$ ; one can simply inspect the form matrix of the last section. This provides the base case for our induction.

Now we proceed to show how the sequence can be extended to give a new sequence satisfying the conditions with  $k$  replaced by  $k + 8$ . We do so using the following form matrix:





Suppose further that the  $n$ th square in the sequence is  $(2, k)$ .

These conditions are similar to the previous ones: the first two conditions ensure an appropriate border, while the second two give “wavefronts” and the last two give squares in front of the waves. The wavefront for condition 3 can be obtained by starting a knight on  $(1, (m+1)/2)$  and moving in direction 5 repeatedly; the wavefront for condition 4 corresponds to a knight which starts on  $(1, k)$  and moves in direction 6 repeatedly.

Further, we can see that our path as established in the previous three steps obeys these conditions with  $k = k_0 = (m+1)/2 + 11$ ; this can be seen by inspection of the previous form matrix. This gives the base case for our induction.

We now exhibit, in Figure 6, a form matrix which will allow us to extend this path to a path satisfying the conditions above with  $k$  replaced by  $k + 4$ . In fact, there are three slightly different methods of obtaining such a path, all depicted in the same form matrix. Specifically, if  $k - k_0 = 4$  (resp.  $k - k_0 = 8$ ,  $k - k_0 = 12$ ), we use the path beginning on the indicated “a” (resp. “e”, “i”) and terminating on the last “d” (resp. “h”, “m”). The reader can check that indeed each such path satisfies the given conditions for an appropriate  $k$ . Note the “bump” in the middle of the board where the pattern shifts slightly, and observe that the three paths vary slightly in the moves and the pattern of  $x$ 's which appear at the leftmost edge of the area traversed:

a-d					e-h					i-l				
x	x	x	a	x	x	x	x	x	e	x	x	x	x	i
x	x	x	c	b	x	x	e	x	-	x	x	i	x	k
x	c	b	-	d	x	x	g	f	-	x	x	k	j	-
x	-	d	-	-	x	f	-	h	-	x	j	-	m	-
d	-	-	-	-	-	h	-	-	-	x	k	-	-	-

These are the only differences between the paths, which outside these areas are identical.

The exhibited paths complete our induction, and therefore we can obtain a path satisfying the conditions for any  $k$  of the form  $(m+1)/2 + 11 + 4t$ , as long as  $k \leq m - 4$ . We therefore conclude step 4 by extending our path to one which satisfies the conditions with  $k = m - 4$  and which ends on the square  $(2, m - 4)$ .

## 5.5 Step 5

Although steps 5 and 6 are very simple, we encounter here a notational difficulty. We are not able to tell which of the three similar paths given in step 4 was executed just before we reached the square  $(2, m - 4)$  on which step 4 terminated. This could make it troublesome to draw a precise form matrix, since we do not know how the left edge should look.

We are saved from this difficulty by a convenient fact: to the left of a certain column, each of the paths will form the following pattern:

```

x  x  a  x
a  x  x  b
x  b  e  d
e  d  c  f
c  f  i  h
i  h  g  j
g  j  -  m
-  m  k  -
k  -  -  -

```

We recognize this pattern as that which was repeated at the right-hand vertical line in the center of Figure 6. Further, we notice that after any of the three paths given in Figure 6 is executed, a pattern like the above is repeated in the center of the board. We conclude that if the reader is willing in an individual instance to determine which of the left-hand shapes to use from the three above, we can without loss of clarity declare the following rule: when we encounter a double vertical line in a form matrix, we continue across this line just as if we had crossed the line to the right of center in Figure 6. We carry on with the pattern found in Figure 6 until we re-cross the line, at which point we resume our place in the given form matrix. Some adjustment may be necessary at the extreme left side of the pattern but this will be clear in any specific case.

With this issue resolved we can now present the form matrix which allows us to pass the upper right corner while continuing our wavefront motion. The path given in the form matrix below begins on the square  $(1, m - 2)$ . This allows us to connect it with the path from step 4, which ended on the square  $(2, m - 4)$ .

x	x	x	x	x	x	x	x	4	1	x	x
x	x	x	x	x	x	11	x	x	x	5	2
x	x	x	x	a	x	x	b	10	3	x	x
x	x	a	x	x	b	d	12	104	101	9	6
a	x	x	b	e	13	105	102	205	7	x	x
x	b	e	d	c	d	i	201	106	103	208	8
e	d	c	f	i	h	107	204	m	206	x	x
c	f	i	h	d	k	-	211	202	209	-	207
i	h	g	j	-	m	203	m	-	m	x	x
g	j	-	m	k	-	-	-	210	-	-	-
-	m	k	-	-	-	m	-	-	-	x	x
k	-	-	-	-	-	-	-	-	-	-	-
-	-	-	-	-	-	-	-	-	-	x	x
-	-	-	-	-	-	-	-	-	-	-	-

This matrix alternates numbers and letters; the numbers less than 100, those between 100 and 200, and those over 200 form the three separate sets of numbers used. So we begin on 1 and travel to 13, which takes us to an "a"; the letters run out at the last b, which is adjacent to 101; at 107 we have no more numbers less than 100 and so travel to a "c"; the letters again run out at the last "h" and

we run from 201 to 211, whereupon we move to an “i” and continue on letters until the last “m” which appears on square  $(9, m - 1)$ .

Adding this path to that already created completes step 5.

## 5.6 Step 6

Again we present an inductive argument that allows us to fill a section of the board; here the region filled will consist of all unvisited squares with the exception of a triangular region in the lower right-hand corner.

Suppose that for some  $n$ , there is an odd  $k \pmod{4}$  such that if  $(i, j)$  is a square in the first  $n$  terms of the sequence with  $i < m - 17$ , then  $i$  and  $j$  satisfy one or more of the following conditions, and suppose further that all such  $(i, j)$  which satisfy at least one of the conditions have been visited.

1.  $i = 1, j \equiv 1$  or  $j \equiv 2 \pmod{4}$
2.  $i = 2, j \equiv 0$  or  $j \equiv 3 \pmod{4}$
3.  $i$  is odd and  $j = m - 2$  or  $j = m - 1$
4.  $i \leq m + 5 - 2j$
5.  $j \leq m - 2i + 2k$
6.  $j = m - 2i + 2k + 1$  and  $j \geq (m + 1)/2 + 4$
7.  $(i, j) \in \{(k + 1, m - 1), (k + 2, m - 2), (k + 3, m), (k + 3, m - 2), (k + 4, m - 2), (k + 4, m - 4), (k + 5, m - 3), (k + 6, m - 5)\}$

Suppose further that the  $n$ th square in the sequence is  $(k + 1, m)$ .

The first three conditions guarantee the border is appropriate; the fourth is the same as the first wavefront condition in step 4; the fifth gives a wavefront which starts on  $(k, m)$  and moves in direction 6 repeatedly; and the last two ensure that certain squares outside the wavefront have been visited.

We can verify that after step 5 we have a path which satisfies these conditions with  $k = 5$ . With the base case in hand, we show a form matrix which gives the inductive step.



x	x	x	x	x	x	x	x	x	x	x
x	x	x	x	x	x	x	x	x	x	x
x	x	x	x	x	a	x	x	x	a	x
x	x	x	a	x	x	x	c	x	x	x
x	a	x	x	b	-	a	x	a	-	a
x	x	b	-	c	x	c	-	c	x	x
b	-	c	b	-	-	-	a	-	-	-
c	b	-	-	-	c	-	-	-	x	x
-	-	-	-	-	-	-	-	-	-	-
-	-	-	-	-	-	-	-	-	x	x
-	-	-	-	-	-	-	-	-	-	-
-	-	-	-	-	-	-	-	-	x	x

Here the double line at the left implies that the pattern continues in that direction; the matrix of Figure 6 makes it plain how this should be done, although we will have to choose the appropriate one of the three left-side arrangements (seen above) when we reach the left side.

We see that we can extend the path to one satisfying the conditions for any  $k$  of the form  $5 + 2t$ . We do so until we reach the square  $((m - 1)/2 + 1, m - 2)$ , where the tiebreak rule changes and we move on to a new step.

### 5.7 Steps 7,8, 9

The remaining steps should establish that the remaining squares in the lower right-hand corner are also filled by the algorithm. However I did not have enough time in the REU program to complete this proof.

This concludes our partial proof of the algorithm's correctness for the case  $m \equiv 7 \pmod{8}$ . The other cases have proofs which are just as lengthy, and so we omit these proofs. For each of these cases we have shown in Figure 4 the progress of the tour generated by the algorithm: the tour first visits and fills the area marked 1, then the area marked 2, and so forth.

## 6 Warnsdorff-Consistent Tours

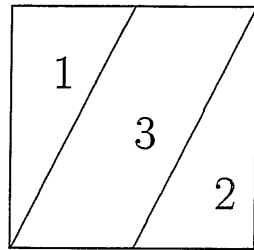
We would like to establish

**Theorem.** For any  $m$  by  $m$  chessboard with  $m \geq 5$ , there exists a tour consistent with Warnsdorff's rule.

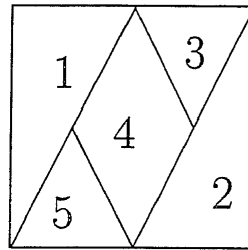
The algorithm presented and partially proven above establishes the theorem for  $m \geq 112$ . To show the theorem is true for  $m < 112$ , we used a computer search for single move-orderings which work on these boards. It turns out that

1. the ordering 21345678 works if

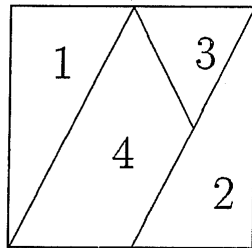
$$m \in \{41, 52, 66, 74, 79, 88, 94, 98, 107, 108\};$$



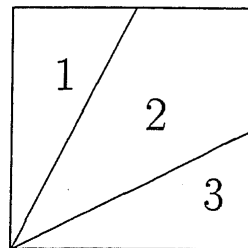
0 mod 8



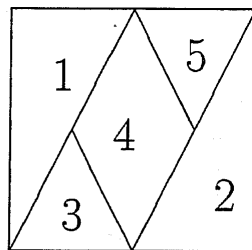
1 mod 8



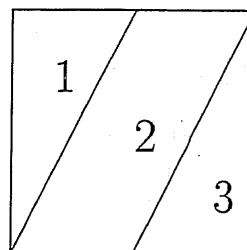
2 mod 8



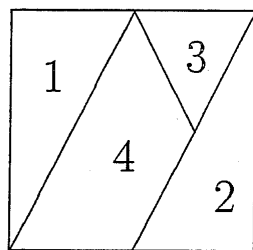
3 mod 8



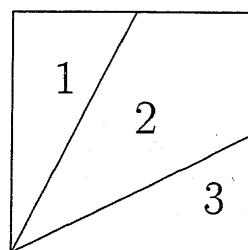
4 mod 8



5 mod 8



6 mod 8



7 mod 8

Figure 4: Summary of tours produced by algorithm.

2. the ordering 13245678 works if  $m = 87$ ;
3. the ordering 12345678 works on all other  $m$  with  $5 \leq m \leq 112$ .

The algorithm presented above can easily be modified to incorporate these observations and thus produce Warnsdorff-consistent tours on all  $m$  by  $m$  boards with  $m \geq 5$ .

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1	4	81	170	175	6	163	208	241	8	161	266	10	159	11	158	12	157	13	156	14	155	15	154	16	153	17	152	18	151	19	150	20	149	21	148	22	147	
2	5	164	209	202	7	162	265	242	9	160	266	10	159	11	158	12	157	13	156	14	155	15	154	16	153	17	152	18	151	19	150	20	149	21	148	22	147	
3	80	169	174	201	176	207	240	229	242	9	160	266	10	159	11	158	12	157	13	156	14	155	15	154	16	153	17	152	18	151	19	150	20	149	21	148	22	147
4	83	178	165	210	203	228	205	264	242	9	160	266	10	159	11	158	12	157	13	156	14	155	15	154	16	153	17	152	18	151	19	150	20	149	21	148	22	147
5	168	173	200	177	206	239	230	243	242	9	160	266	10	159	11	158	12	157	13	156	14	155	15	154	16	153	17	152	18	151	19	150	20	149	21	148	22	147
6	84	179	166	211	184	227	204	263	243	9	160	266	10	159	11	158	12	157	13	156	14	155	15	154	16	153	17	152	18	151	19	150	20	149	21	148	22	147
7	167	178	185	182	199	238	231	244	243	9	160	266	10	159	11	158	12	157	13	156	14	155	15	154	16	153	17	152	18	151	19	150	20	149	21	148	22	147
8	180	85	212	189	226	183	262	237	243	9	160	266	10	159	11	158	12	157	13	156	14	155	15	154	16	153	17	152	18	151	19	150	20	149	21	148	22	147
9	77	186	181	198	213	232	245	237	243	9	160	266	10	159	11	158	12	157	13	156	14	155	15	154	16	153	17	152	18	151	19	150	20	149	21	148	22	147
10	86	197	188	225	190	261	236	237	243	9	160	266	10	159	11	158	12	157	13	156	14	155	15	154	16	153	17	152	18	151	19	150	20	149	21	148	22	147
11	187	76	195	214	233	246	236	237	243	9	160	266	10	159	11	158	12	157	13	156	14	155	15	154	16	153	17	152	18	151	19	150	20	149	21	148	22	147
12	196	87	224	191	260	235	236	237	243	9	160	266	10	159	11	158	12	157	13	156	14	155	15	154	16	153	17	152	18	151	19	150	20	149	21	148	22	147
13	75	194	215	234	247	235	236	237	243	9	160	266	10	159	11	158	12	157	13	156	14	155	15	154	16	153	17	152	18	151	19	150	20	149	21	148	22	147
14	88	223	192	259	216	235	236	237	243	9	160	266	10	159	11	158	12	157	13	156	14	155	15	154	16	153	17	152	18	151	19	150	20	149	21	148	22	147
15	193	74	221	248	216	235	236	237	243	9	160	266	10	159	11	158	12	157	13	156	14	155	15	154	16	153	17	152	18	151	19	150	20	149	21	148	22	147
16	222	89	258	217	216	235	236	237	243	9	160	266	10	159	11	158	12	157	13	156	14	155	15	154	16	153	17	152	18	151	19	150	20	149	21	148	22	147
17	73	220	249	217	216	235	236	237	243	9	160	266	10	159	11	158	12	157	13	156	14	155	15	154	16	153	17	152	18	151	19	150	20	149	21	148	22	147
18	90	257	218	217	216	235	236	237	243	9	160	266	10	159	11	158	12	157	13	156	14	155	15	154	16	153	17	152	18	151	19	150	20	149	21	148	22	147
19	219	72	255	217	216	235	236	237	243	9	160	266	10	159	11	158	12	157	13	156	14	155	15	154	16	153	17	152	18	151	19	150	20	149	21	148	22	147
20	256	91	255	217	216	235	236	237	243	9	160	266	10	159	11	158	12	157	13	156	14	155	15	154	16	153	17	152	18	151	19	150	20	149	21	148	22	147
21	71	254	251	217	216	235	236	237	243	9	160	266	10	159	11	158	12	157	13	156	14	155	15	154	16	153	17	152	18	151	19	150	20	149	21	148	22	147
22	92	254	252	217	216	235	236	237	243	9	160	266	10	159	11	158	12	157	13	156	14	155	15	154	16	153	17	152	18	151	19	150	20	149	21	148	22	147
23	253	70	252	217	216	235	236	237	243	9	160	266	10	159	11	158	12	157	13	156	14	155	15	154	16	153	17	152	18	151	19	150	20	149	21	148	22	147
24	69	93	252	217	216	235	236	237	243	9	160	266	10	159	11	158	12	157	13	156	14	155	15	154	16	153	17	152	18	151	19	150	20	149	21	148	22	147
25	94	68	252	217	216	235	236	237	243	9	160	266	10	159	11	158	12	157	13	156	14	155	15	154	16	153	17	152	18	151	19	150	20	149	21	148	22	147
26	95	95	252	217	216	235	236	237	243	9	160	266	10	159	11	158	12	157	13	156	14	155	15	154	16	153	17	152	18	151	19	150	20	149	21	148	22	147
27	67	95	252	217	216	235	236	237	243	9	160	266	10	159	11	158	12	157	13	156	14	155	15	154	16	153	17	152	18	151	19	150	20	149	21	148	22	147
28	96	66	252	217	216	235	236	237	243	9	160	266	10	159	11	158	12	157	13	156	14	155	15	154	16	153	17	152	18	151	19	150	20	149	21	148	22	147
29	66	97	252	217	216	235	236	237	243	9	160	266	10	159	11	158	12	157	13	156	14	155	15	154	16	153	17	152	18	151	19	150	20	149	21	148	22	147
30	97	66	252	217	216	235	236	237	243	9	160	266	10	159	11	158	12	157	13	156	14	155	15	154	16	153	17	152	18	151	19	150	20	149	21	148	22	147
31	65	98	252	217	216	235	236	237	243	9	160	266	10	159	11	158	12	157	13	156	14	155	15	154	16	153	17	152	18	151	19	150	20	149	21	148	22	147
32	98	64	252	217	216	235	236	237	243	9	160	266	10	159	11	158	12	157	13	156	14	155	15	154	16	153	17	152	18	151	19	150	20	149	21	148	22	147
33	98	98	252	217	216	235	236	237	243	9	160	266	10	159	11	158	12	157	13	156	14	155	15	154	16	153	17	152	18	151	19	150	20	149	21	148	22	147
34	63	102	252	217	216	235	236	237	243	9	160	266	10	159	11	158	12	157	13	156	14	155	15	154	16	153	17	152	18	151	19	150	20	149	21	148	22	147
35	100	104	61	252	217	216	235	236	237	9	160	266	10	159	11	158	12	157	13	156	14	155	15	154	16	153	17	152	18	151	19	150	20	149	21	148	22	147
36	103	62	101	252	217	216	235	236	237	9	160	266	10	159	11	158	12	157	13	156	14	155	15	154	16	153	17	152	18	151	19	150	20	149	21	148	22	147

Figure 5: A path constructed from a form matrix.

